Negation as Failure

Kyle Richardson

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A motivating example

Toy Student Database

```
Student(John)
Student(Mary)
Student(Sue)
Teacher(Julia)

Takes(John,C101)
Takes(Mary,C101)
Takes(Mary,C301)
Takes(Sue,C301)
Teaches(Julia,C101)

Math(C101)
Math(C301)
```

Example queries:

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Example queries: Is John taking C101? yes

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Example queries: What courses are taken by Mary? {C101, C301}

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Example queries: Is John taking C301? as far as we know, no

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Example queries: Is C505 not a math course? as far as we know, yes

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Example queries: Is Julia not a student? as far as we know, yes

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Operationalizing not and as far as we know in traditional databases (Codd, 1970): Assume that all tuples not in the database are false.

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Logic Programming: Not all information can not be read by direct inspection, *non-unit clauses*. (Lloyd, 2012)

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Problem: not/as far as we know, not immediately amenable to classical logic.

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negation as failure: inference rule, *roughly*: derive inference $\neg A$ from failure to prove A.

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negation as failure: The inference rule that makes it possible to infer negative facts from our logic programs!

Problem: Logical Consequence and Negation in Logic

- Any formula α is a **logical consequence** of (or *logically entailed by*) a set of formulas $\Gamma = \{F_1, F_2, ...\}$

$$\Gamma \models \alpha$$

iff there is no interpretation in which Γ is **true** and α is **false**.

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E.g., in $\Gamma = \{\texttt{Student(Mary)}, \texttt{Teacher(Julia)}\}$, we have:
$\Gamma \not\models \neg\texttt{Student(Julia)}$ and $\Gamma \not\models \texttt{Student}(\textit{Julia})$

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database context: We want to conclude $\neg\texttt{Student(Julia)}$, but this is not a **sound** inference w.r.t. classical logic.

Logical Consequence and Negation in Classical Logic

- To convince ourselves of this, we can recall the following correspondence between entailment and satisfiability (Davis et al., 1994)[p.235,Thrm2.1]:

Theorem : $\Gamma = \{F_1, ...F_n\} \models \alpha \Leftrightarrow F_1 \wedge .. \wedge F_n \wedge \neg\alpha$ is **unsatisfiable**

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```
## pip install python-sat
from pysat.solvers import Glucose3
### Gamma = [Student(Mary), Teacher(Julia)],

### Does ~Student(Julia) follow from Gamma?
try1 = Glucose3()
try1.add_clause([1])  ## 1=F_1=Student(Mary)
try1.add_clause([2])  ## 2=F_2=Teacher(Julia)
try1.add_clause([3])  ## 3=alpha=Student(Julia)
### conjunctive formula: 1 & 2 & 3
print(try1.solve())#=> 'sat', i.e., G={1,2} |/= not 3

### Does Student(Julia) follow from Gamma?
try2 = Glucose3()
try2.add_clause([1])  ## 1=F_1=Student(Mary)
try2.add_clause([2])  ## 2=F_1=Teacher(Julia)
try2.add_clause([-3]) ## -3=not Student(Julia)
### conjunctive formula: 1 & 2 & ~3
print(try2.solve())#=> 'sat', i.e., G={1,2} |/=  3
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try1.add_clause([3,-3])  ## 3 can be  true or false

### enumerate all possible interpretations
for interpretation in try1.enum_models():
    print(interpretation)
# [1, 2, -3]
# [1, 2, 3]
# neither 3 nor -3 occurs in every model
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The point: Additional machinery is needed to operationalize not/as far as we know.

Logic Programming: Some Reminders

- **Declarative Semantics** (What?): Based on subsets of first-order logic (FOL), focus here: **Horn subset** and **definite clauses**:

$$\underbrace{A_0}_{\text{head}} \leftarrow \underbrace{A_1, ..., A_n}_{\text{body}} (n \geq 0) \quad \text{(clausal form } \forall \quad \underbrace{A_0}_{\text{pos. literal}} \quad \vee \quad \underbrace{\neg A_1}_{\text{neg. literal}} \quad \vee ... \vee \neg A_n)$$

consisting of a *single* positive literal, no negation. **Definite program** P: any set of definite clauses; **Normal programs**: allow negation in body.

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- **Procedural Semantics** (How?): Want to answer a **goal** query:

$$\leftarrow G_1, ... G_m \qquad \text{(clausal form } \forall \neg G_1 \vee ... \vee \neg G_m)$$

computed using special case of Resolution (Robinson, 1965), **SLD-resolution**. **aim** for program P:

$$P \models \exists (G_1 \wedge ... \wedge G_m) \underbrace{\theta_1 ... \theta_m}_{\text{answers}}$$

Closed World Assumption (CWA): Intuition

- The general form of the closed-world assumption (CWA, (Reiter, 1981)) for any FOL theory Γ:

$$\mathrm{CWA}(\Gamma) = \Gamma \cup \left\{ \neg\alpha \mid \ \alpha \text{ is ground formula s.t. } \Gamma \not\models \alpha \right\},$$

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E.g., in $\Gamma = \{$`Student(Mary)`, `Teacher(Julia)`, `Teaches(Julia,C101)`, `Math(C101)`, `Math-teacher(`X`)` $\leftarrow$ `Math(Y)`, `Teaches(X,Y)`$\}$

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$$\mathrm{CWA}(\Gamma) = \Gamma \cup \{\neg\texttt{Teacher(Mary)}, \neg\texttt{Student(Julia)}, \neg\texttt{Teaches(Mary,C101)}, \neg\texttt{Math-teacher(Mary)}, \ldots\}$$

Closed World Assumption (CWA): Properties and Problems

- An important property of classical logic, **monotonicity**:

Given any theories Γ, Γ' s.t. $\Gamma \subseteq \Gamma'$, forall α : $\quad \Gamma \models \alpha$ implies $\Gamma' \models \alpha$

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This is lost with CWA(**non-monotonic**), e.g.,
$\Gamma = \{\texttt{Student(Mary)}, \texttt{Teacher(Julia)}, ...\}$, $\Gamma' = \Gamma \cup \{\texttt{Teacher(Mary)}\}$
we have that

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- Bigger Problem: FOL (even restricted to horn clauses) is undecidable (i.e., no general algorithm can show $\Gamma \not\models \alpha$ in finite-time), not possible to compute $\mathrm{CWA}(\Gamma)$ from Γ.

Negation as (finite) Failure

- A form of CWA with emphasis on *finite failure*:

$$\mathrm{CWA}_{\text{finite}} = \Gamma \cup \left\{ \neg\alpha \mid \text{ground } \alpha \text{ s.t. all attempts at } \Gamma \models \alpha \text{ (finitely) fail} \right\},$$

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Operationally: *flip* notion of success and failure (Lloyd, 2012); e.g.,
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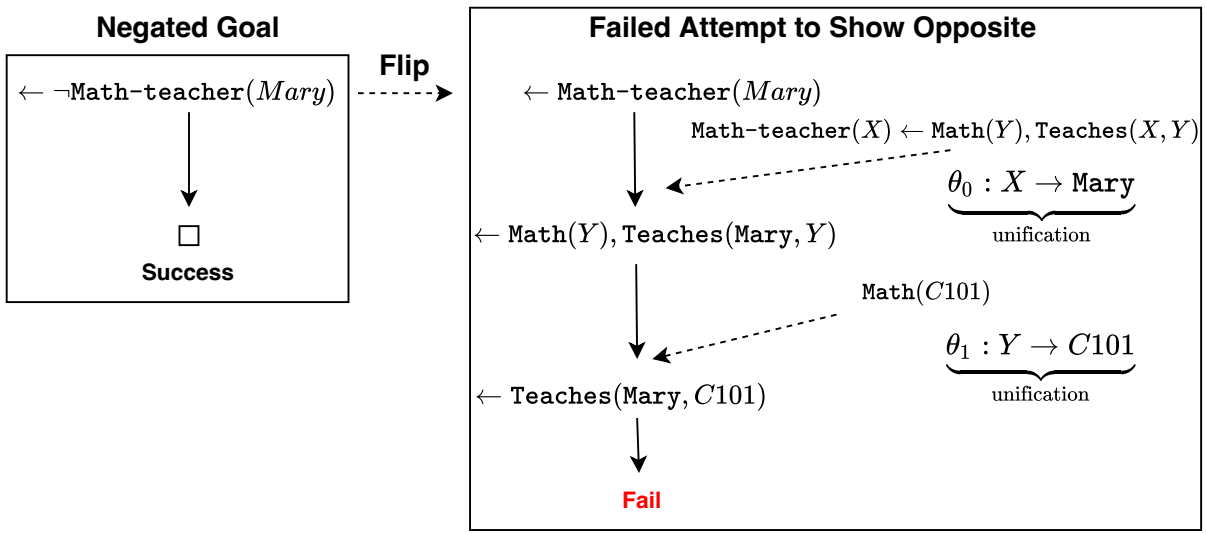

Negation as (finite) Failure: How to operationalize

- **SLD-resolution**: resolution inference rule for *definite programs*:

$$\frac{\overbrace{\leftarrow G_1, \ldots, G_{i-1}, \underbrace{G_i}_{\text{selected}}, \ldots, G_m}^{\text{goal}}\ (m \geq 0) \qquad \overbrace{\underbrace{A_0}_{\text{selected}} \leftarrow A_1, \ldots, A_n}^{\text{selected clause}}\ (n \geq 0)}{\leftarrow G_1, \ldots, G_{i-1}, A_1, \ldots, A_n, \ldots G_m} \quad \underbrace{G_i.\theta = A_0.\theta}_{\text{unifyable}}$$

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coupled with selection and search method; permits inferring *positive information* when repeated application leads to the *empty goal*.

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- **SLDNF-resolution**: SLD-resolution extended with additional rule for *negation as failure*, used in Prolog:

$$\frac{\leftarrow G_1, \ldots, G_{i-1}, \overbrace{\neg G_i}^{\text{negation}}, \ldots, G_m}{G_1, \ldots, G_{i-1}, \ldots, G_m}$$ all attempts at G_i (finitely) fail

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Question: Can we reconcile this with logic, does it make sense?

Negation as (finite) Failure: Semantics

Clark (1978) gave one of the earliest declarative semantics for negation as failure, based on notion of **program completion**.

Keith L. Clark

Department of Computer Science & Statistics

Queen Mary College, London, England

ABSTRACT

A query evaluation process for a logic data base comprising a set of clauses is described. It is essentially a Horn clause theorem prover augmented with a special inference rule for dealing with negation. This is the negation as failure inference rule whereby ~ P can be inferred if every possible proof of P fails. The chief advantage of the query evaluator described is the effeciency with which it can be implemented. Moreover, we show that the negation as failure rule only allows us to conclude negated facts that could be inferred from the axioms of the completed data base, a data base of relation definitions and equality schemas that we consider is implicitly given by the data base of clauses. We also show that when the clause data base and the queries satisfy certain constraints, which still leaves us with a data base more general than a conventional relational data base, the query evaluation process will find every answer that is a logical consequence of the completed data base.

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  E.g., $\Gamma = \{\texttt{Student}(\mathit{Mary}), \texttt{Teacher}(\mathit{Julia})\}$

  $\textsc{Comp}(\Gamma) = \Gamma \cup \{\texttt{Student}(X) \leftrightarrow \underbrace{X = \mathit{Mary}}_{\text{equality pred.}}, \texttt{Teacher}(X) \leftrightarrow \underbrace{\texttt{X = Julia}}_{\text{equality}}\}$

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compl_G.add_clause([1])  ## Gamma = [ 1=Student(Mary),
compl_G.add_clause([2])  ## 2=Teacher(Julia) ]
   ## queries = [ 3=Student(Julia),  4=Teacher(Mary) ]
## 5 = (Mary = Julia) = (Julia = Mary) = False
compl_G.add_clause([-5])
### Student(Julia) <-> Julia=Mary
compl_G.add_clause([-3,5]) # 3 -> 5
compl_G.add_clause([-5,3]) # 5 -> 3
### Teacher(Mary) <-> Mary = Julia
compl_G.add_clause([-4,5]) # 4 -> 5
compl_G.add_clause([-5,4]) # 5 -> 4
for interp in compl_G.enum_models(): print(interp)
# [1, 2, -3, -4, -5]
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compl_G.add_clause([-5,4]) # 5 -> 4
for interp in compl_G.enum_models(): print(interp)
# [1, 2, -3, -4, -5]
```

Result: $\textsc{Comp}(\Gamma) \models \neg \mathit{Student}(\mathit{Julia})$ and $\textsc{Comp}(\Gamma) \models \neg \mathit{Teacher}(\mathit{Mary})$

Negation as (finite) Failure: Semantics

- **Intuition**: When using theory/program Γ with CWA, we *actually* consider the **completion** of Γ, $\textsc{Comp}(\Gamma)$.

E.g., $\Gamma = \{\texttt{Student}(\mathit{Mary}), \texttt{Teacher}(\mathit{Julia})\}$

$$\textsc{Comp}(\Gamma) = \Gamma \cup \{\texttt{Student}(X) \leftrightarrow \underbrace{X = \mathit{Mary}}_{\text{equality pred.}}, \texttt{Teacher}(X) \leftrightarrow \underbrace{\texttt{X = Julia}}_{\text{equality}}\}$$

```
compl_G.add_clause([1])  ## Gamma = [ 1=Student(Mary),
compl_G.add_clause([2])  ## 2=Teacher(Julia) ]
   ## queries = [ 3=Student(Julia),  4=Teacher(Mary) ]
## 5 = (Mary = Julia) = (Julia = Mary) = False
compl_G.add_clause([-5])
### Student(Julia) <-> Julia=Mary
compl_G.add_clause([-3,5]) # 3 -> 5
compl_G.add_clause([-5,3]) # 5 -> 3
### Teacher(Mary) <-> Mary = Julia
compl_G.add_clause([-4,5]) # 4 -> 5
compl_G.add_clause([-5,4]) # 5 -> 4
for interp in compl_G.enum_models(): print(interp)
# [1, 2, -3, -4, -5]
```

Possible to reconcile with classical logic and $\neg$.

Clark Completion Rules

- **Clark Completion** (Clark, 1978): standard set of completion rules coupled with *equality axioms* (special cases excluded).

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  - **1. Variable Restrictions**: For all **facts** $A(t_1, ..., t_o) \leftarrow \underline{\text{add}}$

$$A(x_1, ..., x_o) \leftarrow x_1 = t_2 \wedge ... \wedge x_o = t_o$$

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  - **2. Material Equivalences**: for all **rules** $A(X) \leftarrow A'_i$ $(\forall i \leq k)$ <u>add</u>:

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- **3. Equality**: add $\neg(t_i = t_j)$ for all non-unifiable terms t_i, t_j. **Undefined predicates**: negate any k-ary predicate A in body not included as head:

$$\neg A(X_1, .., X_k)$$

Clark Completion Rules: Example

E.g., $\Gamma = \{\texttt{Student(Mary)}, \texttt{Teacher(Julia)}, \texttt{Teaches(Julia,C301)}, \texttt{Math(C101)}, \texttt{Math-teacher}(X) \leftarrow \texttt{Math}(Y), \texttt{Teaches}(X, Y)\}$

$$\textsc{Comp}(\Gamma) = \Gamma \cup \Big\{$$
$$\texttt{Student}(X) \leftrightarrow \texttt{X=Mary},$$
$$\texttt{Teacher}(X) \leftrightarrow \texttt{X=Julia},$$
$$\texttt{Math}(X) \leftrightarrow \texttt{X=C101},$$
$$\texttt{Math-teacher}(X) \leftrightarrow \texttt{Math}(Y), \texttt{Teaches}(X, Y),$$
$$\texttt{Teaches(X,Y)} \leftrightarrow \texttt{X=Julia}, \texttt{Y=C301},$$
$$\neg(\texttt{Mary} = \texttt{Julia}), \neg(\texttt{Mary} = \texttt{C101}), \neg(\texttt{Julia} = \texttt{C101})$$
$$\Big\}$$

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$$\textsc{Comp}(\Gamma) = \Gamma \cup \Big\{ \begin{array}{l} \texttt{Student}(X) \leftrightarrow \texttt{X=Mary}, \\ \texttt{Teacher}(X) \leftrightarrow \texttt{X=Julia}, \\ \texttt{Math}(X) \leftrightarrow \texttt{X=C101}, \\ \texttt{Math-teacher}(X) \leftrightarrow \texttt{Math}(Y), \texttt{Teaches}(X, Y), \\ \texttt{Teaches(X,Y)} \leftrightarrow \texttt{X=Julia}, \texttt{Y=C301}, \\ \neg(\texttt{Mary} = \texttt{Julia}), \neg(\texttt{Mary} = \texttt{C101}), \neg(\texttt{Julia} = \texttt{C101}) \end{array} \Big\}$$

Note: used as a declarative semantics for negation as failure, can be used directly (**computable**) with general FOL/SAT solvers.

Properties of Clark Completion and Negation as Failure

Some important properties of completion, stated for *definite programs*.

- **Completion extends** Γ, i.e., $\Gamma \models \alpha$ implies $\textsc{Comp}(\Gamma) \models \alpha$ for any α.
- **Adds no new positive information**, i.e., If $\textsc{Comp}(\Gamma) \models \alpha$ holds for an atomic formula α, then $\Gamma \models \alpha$.

Properties of Clark Completion and Negation as Failure

Some important properties of completion, stated for *definite programs*.

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Relating our procedural approach (*SLDNF*-resolution) to Clark completion, sound for COMP w.r.t success and failure (Clark, 1978).

- If α *succeeds* (*precise definition omitted*) from program/theory Γ via SLDNF with answer θ, then $\text{COMP}(\Gamma) \models \alpha\theta$.
- If α *fails* from program/theory Γ via SLDNF, then $\text{COMP}(\Gamma) \models \neg\alpha$.

Conclusion and Summary

- **Problem**: Deriving negative information from logic programs/databases.
  negation as failure: negate things we can't prove, general **closed-world assumption** (CWA), non-monotonic inference rule.

  Negation as (finite) failure: weaker form of CWA, **procedurally**: extension of SLD-resolution, prove negation by exhaustively showing finite failure to prove opposite.

  declarative semantics in terms of **Clark completion**.

Credits and Additional Reading

- Many ideas and examples taken from the following (see for more details):

  Lecture notes: Richard Mayr: https://www.inf.ed.ac.uk/teaching/courses/lp/2014/slides/lpTheory7.pdf, Marin Mircea: https://staff.fmi.uvt.ro/~mircea.marin/lectures/LFP/FoundationsOfLP.pdf, Vladimir Lifshitz https://www.cs.utexas.edu/users/vl/teaching/lbai/completion.pdf, Marek Sergot: https://www.doc.ic.ac.uk/~mjs/teaching/KnowledgeRep491/NBF_491-2x1.pdf

  General references *general logic programming* (Lloyd, 2012), *SLDNF-resolution* (Nienhuys-Cheng and Wolf, 1997), *NAF* (Clark, 1978; Shepherdson, 1998).

Thank you.

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