Introduction to Probability

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Probability and Dealing with Uncertainty

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Features of Questions 2-3: Do not have a yes/no answer, require some **measurement** or **(random) experimentation**.

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One Definition: Probability is a measure of the size of a set.[1]

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A Simple Model of Probability

Components: 1) sample space Ω (set of all possible outcomes); 2) *event* E (some subset of Ω); *probability* (number that describes relative size of E and Ω)

$$\underbrace{\mathcal{P}}_{\text{measure}}\underbrace{[E]}_{\text{set}} = \underbrace{\frac{E}{\Omega}}_{\text{number} \in [0,1]}$$

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2. What is the probability of heads twice $(\mathbf{H}, \mathbf{H})$ after two fair coin flips?

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More complex (random) **"experiment"**, interested in the outcomes of two separate events (aka random variables) (X_1, X_2).

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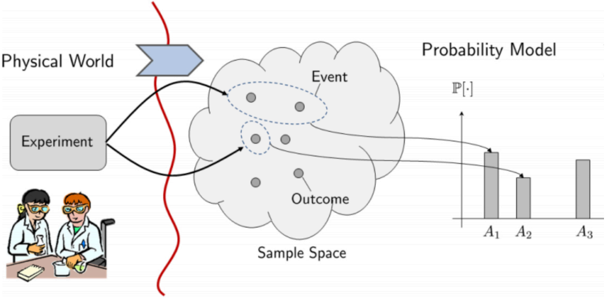
Subjective probability vs. *relative frequentist* interpretation of probability (our mini theory of probability so far).

The General Picture: Elements of a Probabilistic Model



Components: Sample Space, Events, Probability measure $\mathcal{P}$

Mathematical Tools: set theory, combinatorics, (measure theory), ...

Starting Point: Probability is a measure of the size of a set.

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  1. For every event E, $\mathcal{P}[E] \geq 0$. (**non-negativity**)
  2. $\mathcal{P}[\Omega] = 1$
  3. For disjoint events $\{E_1, E_2, ..., E_n\}$ (i.e., where for each E_i, E_j, $E_i \cap E_j = \emptyset$), (**additivity**)

$$\mathcal{P}[E_1 \cup E_2 \cup ... \cup E_n] = \sum_{i=1}^{n} \mathcal{P}[E_i]$$

Consequences of the Axioms

- The probability of events are *less than or equal to* 1, i.e., $0 \leq \mathcal{P}[E] \leq 1$

  Hint: Axiom 1 (the first part of inequality) combined with Axiom 2.

- **monotonicity:** If $E \subseteq E'$ then $\mathcal{P}[E] \leq \mathcal{P}[E']$
- $\mathcal{P}[\emptyset] = 0$

  $\mathcal{P}[E] = \mathcal{P}[E \cup \emptyset]$(since any $E \cup \emptyset = E$) $= \mathcal{P}[E] + \mathcal{P}(\emptyset)$ (axioms 3)

- If $E_1 \cap E_2 = \emptyset$, $\mathcal{P}[E_1 \cup E_2] = \mathcal{P}[A_1] + \mathcal{P}[A_2]$
- For event E and $\mathcal{P}[E]$, $\mathcal{P}[E^c] = 1 - \mathcal{P}[E]$

  Hint: Apply above, $E = E_1$, $E^c = E_2$ since $E \cap E^c = \emptyset$

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> Viewed in this way, we can then start thinking about subjective probabilities (e.g., prob. of Scottish Independence)
> Rules of probability still play a very important role!

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Scenario: [3] We bet on the proposition **Heads** for a stake $S = \$1$, and an agent's *betting quotient* q (amount lost if loss); optionally: s for $\neg$**H**.

General Setting

Lands H	**Payoff**
True	$\$1 - \$q$
False	$-\$q$

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https://plato.stanford.edu/cgi-bin/encyclopedia/archinfo.cgi?entry=dutch-book

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Expected value: $(0.5 * 0.5) + (-0.5 * 0.5) = 0$

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Dutch book argument: making wagers in way that flouts probability axioms (e.g., axiom 2) yields irrational behavior (someone always loses).

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Conclusion

- Probability theory: A mathematical tool we can use to reason about uncertainty; key concepts: probability as a measure of the size of sets, **(random) events**, **sample space**, Kolmogorov axioms.
- Can serve as a good scientific inductive logic, **subjective probability**, *flouting rules can lead you astray*, (standard) probability theory matters!

Thank you.

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